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Model predictive control using reduced order models: Guaranteed stability for constrained linear systems☆

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ABSTRACT

The problem of controlling a high-dimensional linear system subject to hard input and state constraints using model predictive control is considered. Applying model predictive control to high-dimensional systems typically leads to a prohibitive computational complexity. Therefore, reduced order models are employed in many applications. This introduces an approximation error which may deteriorate the closed loop behavior and may even lead to instability. We propose a novel model predictive control scheme using a reduced order model for prediction in combination with an error bounding system. We employ the explicit time and input dependent bound on the model order reduction error to achieve design conditions for constraint fulfillment, recursive feasibility and asymptotic stability for the closed loop of the model predictive controller when applied to the high-dimensional system. Moreover, for a special choice of design parameters, we establish local optimality of the proposed model predictive control scheme. The proposed MPC approach is assessed via examples demonstrating that a good trade-off between computational efficiency and conservatism can be achieved while guaranteeing constraint satisfaction and asymptotic stability.

1. Introduction

In process control, systems are frequently described by high-dimensional linear system with hard input and state constraints. We consider the problem of applying model predictive control (MPC) to this systems. Examples of constrained high-dimensional linear system which may be controlled by MPC are an industrial glass feeder [1], a distributed reactor model [2], or a large-scale flow [3]. The high-dimensionality of the system leads to a large computational burden which is prohibitive for MPC. Therefore, reduced order models (ROMs) [4,5] are frequently used for the prediction in MPC [1–3,6–16]. Linear ROMs are exploited in MPC to facilitate the solution of the online optimization problem [1,2]. For explicit MPC formulations either ROMs or a projection of the state may be used to reduce the number of regions [3,17].

By using an ROM for prediction, a mismatch between the high-dimensional system and the prediction model is introduced. As a result, satisfaction of constraints or asymptotic stability of the closed loop may be lost. These are important issues especially for safety critical applications. Hence, robustness against the model order reduction (MOR) error has to be ensured by the MPC scheme.

This question has already been considered in literature. In [15], the authors propose to employ the a priori error bound of balanced truncation to tighten the output constraints. However, recursive feasibility and asymptotic stability are not established in [15]. When the high-dimensional system is given by a parabolic partial differential equation (PDE), asymptotic stability of the closed loop system and hard state constraint satisfaction can be guaranteed [16]. In [16], the MOR method is limited to modal decomposition, where the states of the ROM and the neglected dynamic are only coupled by the input. This allows to decouple the unstable modes

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from the MOR error resulting in a stable error system. Then, input-to-state boundedness of the error system is exploited to establish satisfaction of hard state constraints. However, in [16] the state constraints are tightened according to a bound on the error between the high-dimensional system and the ROM. To predict this bound, all inputs within the input constraints are taken into account. Thereby, significant conservatism is introduced, as this comparatively large error bounding sets are used to tighten the constraints. Hence, an error bound that takes the predicted input trajectory into account may be significantly tighter.

Alternatively, methods from robust MPC [18,19] could be used in principle. As was noted in [3], "the applicability of these methods to establish robustness in the context of MPC with reduced-order models remains a challenging open research question". In the meantime, results of robust output feedback MPC have been specialized to guarantee robust stability despite the MOR error [20]. The results in [20] do not rely on an explicit bound on the MOR error and, hence, apply only to stable systems and furthermore soft state constraints. In the recent work [21] it is suggested to employ methods from robust MPC to establish asymptotic stability of a possibly large set around the origin. However, asymptotic stability of the origin is not established in [21]. Furthermore, similar to [16], the input trajectory is not taken into account in the prediction of the uncertainty induced by the MOR.

To the best of our knowledge, satisfaction of hard state constraints and asymptotic stability of the origin for MPC using an ROM is established, only in [16]. However, as discussed above, in [16] the MOR method is limited to modal decomposition.

We present a new approach to MPC using an ROM with guaranteed asymptotic stability of the origin and constraint satisfaction that allows for many commonly used MOR methods. Our approach is inspired by [22], where a method for dynamical optimization using ROMs is presented, which minimizes an upper bound of the true objective function. To ensure a guaranteed objective function value, an a posteriori bound for the MOR error is employed in [22]. This error bound, originally suggested for parametrized systems in [23], is given by a scalar ordinary differential equation (ODE). This results in an error bound that takes the input trajectory into account which may be less conservative than the bounds used in [16,21].

In [22], asymptotic stability of the high-dimensional system is required to obtain an asymptotically stable error bounding system. The first contribution of this work is to achieve an asymptotically stable error bounding system also for unstable linear systems by exploiting a feedback of the MOR error. In contrast to [16], we do not require to decouple the unstable modes from the error dynamics to get a finite bound on the MOR error. As a result, general Petrov–Galerkin projection based MOR methods can be used. The flexibility in the MOR method in general results in ROMs with smaller dimension or approximation error compared to modal decomposition.

The second contribution is to employ this error bounding system to establish a novel MPC scheme using an ROM. Our MPC scheme guarantees satisfaction of hard input and state constraints for the controlled high-dimensional system. Furthermore, we derive clear design conditions that guarantee recursive feasibility and asymptotic stability despite the MOR error. Unlike existing MPC approaches using an ROM with guaranteed asymptotic stability of the origin, our approach is applicable to unstable systems with hard state constraints and general Petrov–Galerkin projection based MOR methods. Furthermore, the proposed MPC scheme is locally optimal when the feedback of the MOR error is determined by the optimal solution of the unconstrained case.

In order to assess basic properties of the proposed MPC scheme, we compare the performance and region of attraction of different MPC schemes based on an illustrative academic example. The system depends on a parameter which allows to study the proposed MPC scheme for varying influence of the MOR error. Furthermore, we apply the proposed MPC scheme to a model of a tubular reactor. Thereby, we assess the computational efficiency and conservatism for a practically motivated example.

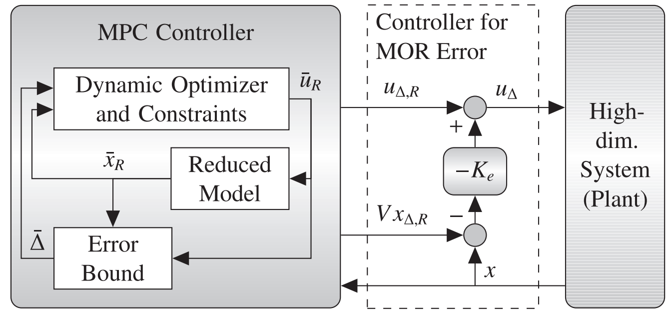

Fig. 1. Structure of the proposed MPC scheme. The overall controller consist of the model predictive controller (Section 6) and the controller for the MOR error (Section 5).

The structure of the proposed MPC scheme is illustrated in Fig. 1. The model predictive controller includes the ROM as well as the scalar error bounding system. The overall controller additionally contains the feedback of the MOR error. For the details of Fig. 1 we refer to the subsequent sections.

The remainder of this work is organized as follows. In the following section, we formally describe the considered problem setup. In Section 3, we introduce the MOR framework. As a starting point for the derivation of the proposed MPC scheme with robustness against the MOR error, we introduce in Section 4 an MPC scheme using an ROM that neglects the MOR error. In Section 5, we show how the MOR error can be bounded by an asymptotically stable system even for unstable high-dimensional systems. This error bounding systems is employed in Section 6 to derive the proposed MPC scheme. Subsequently, we show that constraint satisfaction, recursive feasibility and asymptotic stability despite the MOR error are given if easily verifiable design criteria are fulfilled. In Section 7, we compare different MPC schemes for two examples before we conclude the paper in Section 8.

Notation

We denote the 2-norm of a vector $x \in \mathbb{R}^n$ with $\|x\|$ and the induced 2-norm of a matrix $A \in \mathbb{R}^{n\times m}$ with $\|A\|$. For a matrix A, $\lambda_{\max}(A)$ is the largest real part of all eigenvalues, $\sigma_{\max}(A)$ is the maximal singular value, and $\sigma_{\min}(A)$ is the minimal singular value. We use $A \succ 0$ to denote that the matrix $A \in \mathbb{R}^{n\times n}$ is symmetric and positive-definite. The natural numbers including zero are $\mathbb{N}_0$. The notation I represents the identity matrix. The transpose of the matrix A is denoted by $A^\top$. When writing $x(\cdot)$ we refer to the trajectory of x. The space of square-integrable vector functions on $[0, \infty)$ is represented by $\mathcal{L}_2$. The $\mathcal{L}_2$-norm of $x(\cdot) \in \mathcal{L}_2$ is indicated by $\|x(\cdot)\|_{\mathcal{L}_2}$. Finally, $\mathcal{PC}(\mathbb{A}, \mathbb{B})$ denotes the set of all piece-wise continuous functions $f : \mathbb{R} \supseteq \mathbb{A} \to \mathbb{B}$.

2. Problem setup

We consider continuous-time linear time-invariant high-dimensional systems

$$\Sigma_D : \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{1}$$

in which $x(t) \in \mathbb{R}^n$ is the state vector at time t, $n \gg 1$, $u(t) \in \mathbb{R}^m$ is the input vector at time t, $u(\cdot) \in \mathcal{L}_2$ and piece-wise continuous. In the following Σ_D is called detailed system. The pair (A, B) is assumed to be stabilizable. The states and inputs are constrained by

$$c_k^\top \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \le d_k, \quad k = 1, \ldots, n_C, \tag{2}$$

with $c_k \in \mathbb{R}^{n+m}$ and $d_k \in \mathbb{R}$. For technical reasons, we state the following common assumption.

Assumption 1. The state and input constraints (2) define a compact set and the equilibrium $x = 0$ for $u = 0$ is contained in its interior.

Given the initial state x_0, we want to steer the system to the origin close to optimality with respect to the cost functional

$$J_\infty(x_0, u) := \int_0^\infty F(x(t), u(t))\, dt,$$

subject to the system dynamics (1) as well as the state and input constraints (2). The quadratic stage cost is defined by

$$F(x(t), u(t)) := x^\top(t)Qx(t) + u^\top(t)Ru(t), \tag{3}$$

with Q and R satisfying the following assumption.

Assumption 2. The matrices Q and R are symmetric and positive definite.

We assume that the full state of the detailed system can be measured. This assumption may be relaxed as discussed later in Section 6.3.

The given control problem without the constraints (2) can be solved using the well known linear-quadratic regulator (LQR) [24]. Due to the constraints it is significantly harder to obtain a good performance of the controlled system. Thus, we seek an approximate solution via MPC. For an introduction to constrained MPC we refer to [25,26].

MPC tackles the infinite horizon control problem by solving at every sampling instant an open-loop optimal control problem over a finite horizon T. To solve the optimal control problem, the detailed system is frequently replaced by an ROM as discussed in Section 1. Therefore, we briefly introduce the underlying MOR framework in the following section.

3. Model order reduction

In this work, MOR based on a Petrov–Galerkin projection is considered. For a Petrov–Galerkin projection, the MOR method yields matrices V, W that satisfy the following assumption.

Assumption 3. The matrices $V, W \in \mathbb{R}^{n\times n_R}$ with $n_R \ll n$ fulfill $W^\top V = I$.

Multiplying (1) from left with $W^\top$

$$\begin{aligned} W^\top \dot{x}(t) &= W^\top Ax(t) + W^\top Bu(t), \\ &= W^\top AVx_R(t) + W^\top Ae(t) + W^\top Bu(t), \end{aligned}$$

neglecting the MOR error $e(t) = x(t) - Vx_R(t)$ and replacing as additional degree of freedom u with a new input variable u_R yields the ROM

$$\Sigma_R : \dot{x}_R(t) = A_R x_R(t) + B_R u_R(t), \quad x_R(0) = x_{R,0}, \tag{4}$$

with $A_R = W^\top AV$ and $B_R = W^\top B$. The initial condition of the ROM is given by $x_{R,0} = W^\top x_0$. We emphasize that $x_R(t) = W^\top x(t)$ does not hold in general since the MOR error is neglected. An estimate for the detailed state is $Vx_R(t)$.

The results of this work apply to any MOR method that results in a Petrov–Galerkin projection. Consequently, many of the linear MOR methods may be used, e.g., all methods mentioned in [5]. For the MPC scheme proposed in this work, variations of MOR methods for optimal control based on the proper orthogonal decomposition [27,28] may be a good choice. For clarity and limited amount of space, we focus on MPC using ROMs in this work and leave out the MOR aspect. Therefore, we assume that the matrices W and V required for the Petrov–Galerkin projection are known or computed beforehand.

For the subsequent MPC schemes we assume:

Assumption 4. The pair (A_R, B_R) is stabilizable.

4. Common approach for model predictive control using a reduced order model

As a basis for our approach and the subsequent analysis we introduce in this section an MPC setup that employs only the ROM.

When using Σ_R for prediction, the detailed state is unknown and the estimate $Vx_R(t)$ is used in the stage cost

$$\begin{aligned} F_R(x_R(t), u_R(t)) :&= F(Vx_R(t), u_R(t)) \\ &= x_R^\top(t)Q_R x_R(t) + u_R^\top(t)Ru_R(t), \end{aligned}$$

in which $Q_R = V^\top QV$ is positive definite because $\operatorname{rank}(V) = n_R$.

The following ROM-MPC setup emerges from a common MPC formulation with a terminal cost and a terminal constraint [29, p. 142] by replacing Σ_D with Σ_R and $x(t)$ with $Vx_R(t)$. As a result, at the sampling instant $t_i \ge 0$ the state $x(t_i)$ is measured and the following finite horizon optimal control problem is solved.

Problem 1. (Optimization problem for MPC using the ROM)

$$\underset{\bar{u}_R \in \mathcal{PC}([t_i, t_i+T], \mathbb{R}^m)}{\text{minimize}} \quad J_R(x(t_i), \bar{u}_R)$$

in which

$$J_R(x(t_i), \bar{u}_R) = \int_{t_i}^{t_i+T} F_R(\bar{x}_R(t;t_i), \bar{u}_R(t))\, dt + E_R(\bar{x}_R(t_i+T;t_i)),$$

subject to

$$\dot{\bar{x}}_R(t;t_i) = A_R\bar{x}_R(t;t_i) + B_R\bar{u}_R(t),$$

$$\bar{x}_R(t_i;t_i) = W^\top x(t_i),$$

$$c_k^\top \begin{bmatrix} V\bar{x}_R(t;t_i) \\ \bar{u}_R(t) \end{bmatrix} \le d_k, \quad k = 1, \ldots, n_C, \tag{5a}$$

$$\bar{x}_R(t_i+T;t_i) \in \Omega_R, \tag{5b}$$

for all $t \in [t_i, t_i + T]$.

We denote by $\bar{x}_R(\cdot\,;t_i)$ the predicted trajectory starting from the initial condition $\bar{x}_R(t_i;t_i) = W^\top x(t_i)$ and driven by $\bar{u}_R(t)$ for $t \in [t_i, t_i + T]$. The terminal cost $E_R(\bar{x}_R(t_i + T;t_i))$ and the terminal set Ω_R are additional design variables. They are used in MPC to guarantee asymptotic stability of the closed loop, see, e.g., [26, Theorem 2.1]. The initial condition of the ROM is the projection of the measured detailed state. Furthermore, the state and input constraints (2) are approximated using the estimated detailed state $V\bar{x}_R(t;t_i)$.

Problem 1 describes a common approach in MPC using ROMs, found similarly for discrete-time systems in [2,3,13,15]. The cited work focus on different aspects and therefore, among others, differences in the terminal set and the initial condition of the ROM exist.

We assume that the input trajectory which solves Problem 1 is given by $u_R^*(t;t_i)$ with associated predicted state trajectory $x_R^*(t;t_i)$, $t \in [t_i, t_i + T]$. We distinguish between internal predicted variables, $\bar{x}_R(t;t_i)$, $\bar{u}_R(t)$, $x_R^*(t;t_i)$, $u_R^*(t;t_i)$ and variables appearing outside of the model predictive controller, e.g., the input applied to the detailed system as defined in the following.

The optimal input trajectory $u_R^*(\cdot; t_i)$ is applied to the system until the next sampling instant t_{i+1}

$$u(t) = u_{\mathcal{R}}(t) := u_R^*(t; t_i), \quad t_i \leq t < t_{i+1}. \tag{6}$$

To avoid confusion with variables appearing in the ROM, we use the calligraphic index $\mathcal{R}$ to denote that the input is based on the common ROM-MPC scheme. At t_{i+1}, the state $x(t_{i+1})$ is measured and Problem 1 is solved for the shifted horizon $[t_{i+1}, t_{i+1}+T]$. Repeating this procedure, the input of the detailed system $u(t) = u_{\mathcal{R}}(t)$ for $t \in [0, \infty)$ is defined by the optimal solution of Problem 1 at all sampling instants. For a given sampling time $\delta < T$, we consider equidistant sampling instants $t_i = i\delta$, $i \in \mathbb{N}_0$. If the detailed system is used for prediction, which corresponds to $W = V = I$, we will denote the input applied to the detailed system with $u_{\mathcal{D}}(\cdot)$ instead of $u_{\mathcal{R}}(\cdot)$.

For the application of a model predictive controller and to prove asymptotic stability of the closed loop, it is important that the underlying optimization problem is feasible for all time instants. This would be guaranteed if (i) a feasible solution at $t_0 = 0$ exists and (ii) when applying the model predictive controller from feasibility at $t_0 = 0$ follows feasibility for all subsequent sampling instants. Property (ii) is known as recursive feasibility.

By using Σ_R for prediction, the MOR error is introduced within the MPC scheme. Thus, guarantees for the closed loop with $u_{\mathcal{D}}(\cdot)$ do not necessarily hold any more. For instance, we cannot conclude from the state and input constraints (5a) that the constraints (2) are satisfied. Furthermore, since $W^\top x(t_{i+1}) \neq \bar{x}_R(t_{i+1}; t_i)$, recursive feasibility of Problem 1 and asymptotic stability of the closed loop with $u_{\mathcal{R}}(\cdot)$ have to be addressed.

To enforce the constraints (2), we will derive an upper bound for the error between the state of the detailed system and the ROM. Given the state of the ROM and the error bound, a set containing the detailed state is known. Thus, (2) can be enforced. Furthermore, we use the error bound introduced in the following section to show recursive feasibility and to prove asymptotic stability for an ROM-MPC scheme in Section 6.

5. Asymptotically stable error bounding system

In this section, we derive a bound for the MOR error. The error bound is based on the a posteriori error estimator [23] and its improved version [22]. These previous results require a stable detailed system to derive a stable error bounding system. These results are extended to the case of unstable detailed systems, while preserving the stability of the error bounding system.

To derive the error bound, consider the error

$$e(t) := x(t) - Vx_R(t)$$

between the actual detailed state $x(t)$ and the estimate $Vx_R(t)$ based on the reduced state $x_R(t)$. The error dynamics are given by

$$\begin{aligned}\dot{e}(t) &= \dot{x}(t) - V\dot{x}_R(t) \\ &= Ae(t) + (AV - VA_R)x_R(t) + Bu(t) - VB_R u_R(t).\end{aligned}$$

Due to the term $Ae(t)$, the autonomous error dynamics are determined by the possibly unstable detailed system. To modify the error dynamics, we exploit that the detailed state and thus the MOR error is known. Hence, we can introduce a feedback of the MOR error

$$u(t) = u_R(t) - K_e e(t). \tag{7}$$

The first part $u_R(t)$ will be determined by the model predictive controller to satisfy the state and input constraints (2). The second part $-K_e e(t)$ may keep the MOR error small at the expense of an additional excitation. This idea is similar to tube-based MPC, where a local controller is used to keep the error around a nominal trajectory bounded [18]. Using (7) and defining $A_e := A - BK_e$, the error dynamics are

$$\begin{aligned}\dot{e}(t) &= A_e e(t) + (AV - VA_R)x_R(t) + (B - VB_R)u_R(t) \\ &= A_e e(t) + (I - VW^\top)(AVx_R(t) + Bu_R(t)).\end{aligned} \tag{8}$$

Based on the error dynamics, we obtain the following bound for the error.

Theorem 1. *(Error bounding system) Consider the detailed system Σ_D and the ROM Σ_R defined by V, W. Then, if the feedback matrix K_e is such that $A_e = A - BK_e$ is Hurwitz, there exist $\alpha \geq 1$ and $\beta > 0$ such that the MOR error is bounded by*

$$\forall t \geq 0 : \|x(t) - Vx_R(t)\| \leq \Delta(t),$$

in which $\Delta(t)$ is defined by the error bounding system

$$\Sigma_\Delta : \begin{cases} \dot{\Delta}(t) = -\beta\Delta(t) + \alpha\|r(t)\|, \\ \Delta(0) = \alpha\|x(0) - Vx_R(0)\|, \end{cases} \tag{9}$$

with the residual $r(t) = (I - VW^\top)(AVx_R(t) + Bu_R(t))$.

Proof. The proof is similar to the proof of [22, Theorem 1]. The error dynamics (8) have the solution

$$e(t) = \exp(A_e t)e(0) + \int_0^t \exp(A_e(t-\tau))r(\tau)d\tau.$$

By applying the norm and employing the sub-multiplicative property we obtain

$$\|e(t)\| \leq \|\exp(A_e t)\|\|e(0)\| + \int_0^t \|\exp(A_e(t-\tau))\|\|r(\tau)\|d\tau.$$

Since A_e is Hurwitz, there exist $\alpha \geq 1$ and $\beta > 0$, such that [30, Lemma 3.3.19]

$$\forall t \geq 0 : \|\exp(A_e t)\| \leq \alpha\exp(-\beta t). \tag{10}$$

Hence, the error is bounded from above $\|e(t)\| \leq \Delta(t)$ by

$$\Delta(t) = \alpha\exp(-\beta t)\|e(0)\| + \alpha\int_0^t \exp(-\beta(t-\tau))\|r(\tau)\|d\tau.$$

To conclude the proof, we observe that the error bound $\Delta(t)$ is the solution of the error bounding system (9). □

The parameters α and β in Theorem 1 directly influence the tightness of the error bound. Therefore, an important aspect is the computation of α and β such that (10) holds. In general, there is a trade-off between a small $\alpha \geq 1$ and a large exponential decay rate β. If we choose $\alpha = 1$, then the exponential decay rate β is given by $\beta = -\lambda_{\max}(A_e + A_e^\top)/2$ [30, Lemma 5.5.11], which can be smaller than zero. Alternatively, one can choose $0 < \beta < -\lambda_{\max}(A_e)$ and any positive definite and symmetric $Q \in \mathbb{R}^{n\times n}$. Then, $\alpha = \sqrt{\sigma_{\max}(P)/\sigma_{\min}(P)}$ is determined by the positive definite and symmetric solution $P \in \mathbb{R}^{n\times n}$ of $PA_e + A_e^\top P + Q + 2\beta P = 0$ [30, p. 665]. Since the pair (A, B) is stabilizable, at least one K_e exists such that $0 < -\lambda_{\max}(A_e)$. Thus, $\beta > 0$ can always be achieved. How to obtain a feedback matrix K_e with bounded norm using linear matrix inequalities is shown in [31].

For details concerning the computational efficiency of the error bound (9) and consequences for the MOR method we refer to [22,23]. There, it is also shown for examples that a reasonably tight error bound can be achieved by using an appropriate order for the ROM.

The error bound derived in Theorem 1 gives a relation between the state of the ROM and the state of the detailed system. More precisely, for given state of the ROM and error bound, the set of possible detailed states is given by

$$\mathcal{X}(x_R(t), \Delta(t)) := \{x \in \mathbb{R}^n \mid \|x - Vx_R(t)\| \leq \Delta(t)\}.$$

6. Model predictive control using the reduced order model with guaranteed stability

In Section 4, we have already pointed out that the main drawbacks of the common ROM-MPC approach described by Problem 1 are that neither satisfaction of the state constraints (2) nor asymptotic stability is guaranteed. To overcome these issues, we will employ the error bound from Section 5 and the set of possible detailed states $\mathcal{X}(x_R(t), \Delta(t))$.

6.1. Enforcing state and input constraints

To enforce the state and input constraints (2) we extend the ROM-MPC scheme given by Problem 1 with the error bounding system Σ_Δ of Theorem 1. Based on Σ_Δ, we know the set of possible detailed states $\mathcal{X}(\bar{x}_R(t; t_i), \overline{\Delta}(t; t_i))$ which allows us to enforce the state and input constraints for the detailed system (2) by replacing (5a) with

$$c_k^\top \begin{bmatrix} x(t) \\ \bar{u}_R(t) - K_e(x(t) - V\bar{x}_R(t; t_i)) \end{bmatrix} \leq d_k, \quad k = 1, \ldots, n_C \tag{11}$$

for all $x(t) \in \mathcal{X}(\bar{x}_R(t; t_i), \overline{\Delta}(t; t_i))$.

These constraints can be formulated as polytopic constraints in $\bar{x}_R(t; t_i)$, $\bar{u}_R(t)$, and $\overline{\Delta}(t; t_i)$, i.e., without the set $\mathcal{X}$, as shown by the following proposition.

Proposition 1. *(Reformulation of constraints)*
The constraints (11) are equivalent to

$$c_k^\top \begin{bmatrix} V\bar{x}_R(t; t_i) \\ \bar{u}_R(t) \end{bmatrix} \leq d_k - \overline{\Delta}(t; t_i) \left\| c_k^\top \begin{bmatrix} I \\ -K_e \end{bmatrix} \right\|, k = 1, \ldots, n_C. \tag{12}$$

Proof. The constraints (11) are equivalent to: For every $k = 1, \ldots, n_C$,

$$c_k^\top \begin{bmatrix} V\bar{x}_R(t; t_i) \\ \bar{u}_R(t) \end{bmatrix} + c_k^\top \begin{bmatrix} I \\ -K_e \end{bmatrix} e_k(t) \leq d_k \tag{13}$$

for all $e_k(t) \in \mathbb{R}^n$ with $\|e_k(t)\| \leq \overline{\Delta}(t; t_i)$.

For every k with $\begin{bmatrix} I & -K_e^\top \end{bmatrix} c_k = 0$, equivalence of (12) and (13) directly follows. For all other k, from the Cauchy–Schwarz inequality follows that the left side of (13) is maximized if $e_k(t)$ and $\begin{bmatrix} I & -K_e^\top \end{bmatrix} c_k$ are linearly dependent resulting in

$$e_k(t) = \frac{\overline{\Delta}(t; t_i)}{\left\| [I - K_e^\top] c_k \right\|} [I - K_e^\top] c_k.$$

The proposition follows by inserting this error into (13). □

Up to now, we are able to satisfy the state and input constraints (2) while using the ROM. In the following, we will present intermediate results which are employed afterwards to prove asymptotic stability.

6.2. Terminal cost and terminal set

In this subsection, we define the terminal set based on a terminal controller such that, first, the terminal set is positively invariant for the ROM under the terminal controller and, second, the constraints are satisfied within the terminal set when using this terminal controller. These properties will be used in Section 6.3 to show recursive feasibility. Furthermore, we define the terminal cost which will be used in Section 6.4 to show asymptotic stability.

Since we want to guarantee satisfaction of the constraints, we include the error bound in the definition of the terminal set Ω. To prove invariance of Ω, the following assumption is used below.

Assumption 5. The error bounding system Σ_Δ is asymptotically stable, i.e., $\beta > 0$ and α are chosen such that (10) holds.

As shown in Section 5, Assumption 5 can always be achieved by a feedback of the MOR error.

The following lemma states a possible choice for the terminal set.

Lemma 1. *Consider the ROM Σ_R defined by V, W and the error bounding system Σ_Δ with feedback matrix K_e. Suppose that Assumptions 1–5 hold. Then, there exist $\gamma_1 > 0$, $\gamma_2 > 0$, $P_\Omega \succ 0$, Q_Ω, and K_Ω which determine the terminal set*

$$\Omega := \left\{(x_R, \Delta) \in \mathbb{R}^{n_R} \times \mathbb{R} \left| x_R^\top P_\Omega x_R \leq \gamma_1 \right., 0 \leq \Delta \leq \gamma_2 \right\}, \tag{14}$$

and terminal cost $E(x_R) = x_R^\top Q_\Omega x_R$ such that

(i) *the terminal set Ω is positively invariant for the interconnection of Σ_R and Σ_Δ under the control law $u_R(t) = -K_\Omega x_R(t)$,*
(ii) *for all $(x_R, \Delta) \in \Omega$ the state and input constraints (12) are satisfied for $u_R(t) = -K_\Omega x_R(t)$, and*
(iii) *for all $(x_R, \Delta) \in \Omega$ the closed loop with the ROM and the control law $u_R(t) = -K_\Omega x_R(t)$ satisfies*

$$\dot{E}_R(x_R) + F_R(x_R, -K_\Omega x_R) \leq 0. \tag{15}$$

Proof. Since Assumptions 2 and 4 hold, we know that there exists a unique solution $P_R \succ 0$ of the algebraic Riccati equation [24]

$$P_R A_R + A_R^\top P_R - P_R B_R R^{-1} B_R^\top P_R + Q_R = 0. \tag{16}$$

For the proof of Lemma 1, consider the LQR as terminal controller $K_\Omega = R^{-1} B_R^\top P_R$ and $P_\Omega = Q_\Omega = P_R$.

(i) The ROM Σ_R is independent of the error bound $\Delta(t)$. Since we use the LQR as terminal controller, the set $\Omega_R := \{x_R \in \mathbb{R}^{n_R} | \ x_R^\top P_\Omega x_R \leq \gamma_1\}$ is positively invariant for Σ_R with $u_R(t) = -K_\Omega x_R(t)$. Hence, for all $x_R(t) \in \Omega_R$,

$$\begin{aligned} \dot{\Delta}(t) &\overset{(9)}{=} -\beta\Delta(t) + \alpha \|(I - VW^\top)(AV - BK_\Omega)x_R(t)\| \\ &\leq -\beta\Delta(t) + \alpha\|(I - VW^\top)(AV - BK_\Omega)P_\Omega^{-1/2}\| \|P_\Omega^{1/2} x_R(t)\| \\ &\leq -\beta\Delta(t) + \alpha\sqrt{\gamma_1} \|(I - VW^\top)(AV - BK_\Omega)P_\Omega^{-1/2}\|. \end{aligned}$$

Choosing γ_1 and γ_2 such that

$$\alpha\sqrt{\gamma_1} \|(I - VW^\top)(AV - BK_\Omega)P_\Omega^{-1/2}\| \leq \beta\gamma_2 \tag{17}$$

yields

$$\dot{\Delta}(t) \leq -\beta\Delta(t) + \beta\gamma_2.$$

Thus, for all $\Delta(t) > \gamma_2$, we have $\dot{\Delta}(t) < 0$ and $\Delta(t) = \gamma_2$ results in $\dot{\Delta}(t) \leq 0$. In addition, from (9) follows $0 \leq \Delta(t)$. Consequently, *(i)* holds.

(ii) Applying the norm to each element of (12) and using the triangle inequality yields

$$\begin{aligned} &\left\| c_k^\top \begin{bmatrix} V \\ -K_\Omega \end{bmatrix} x_R + \Delta \left\| c_k^\top \begin{bmatrix} I \\ -K_e \end{bmatrix} \right\| \right\| \\ &\leq \left\| c_k^\top \begin{bmatrix} V \\ -K_\Omega \end{bmatrix} P_\Omega^{-1/2} \right\| \left\| P_\Omega^{1/2} x_R \right\| + \Delta \left\| c_k^\top \begin{bmatrix} I \\ -K_e \end{bmatrix} \right\|, \end{aligned}$$

$k = 1, \ldots, n_C$.

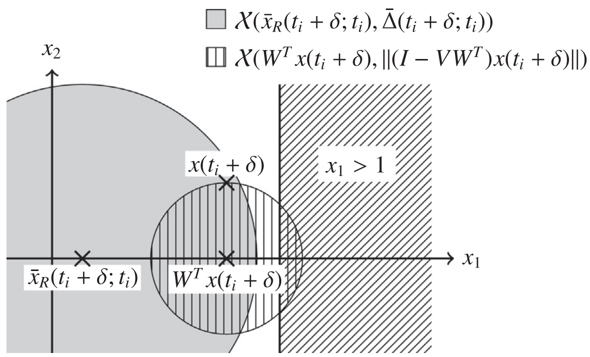

Fig. 2. Illustration of the possible sets for the detailed states for $n=2$, $V=W=[1\ 0]^\top$, and $\alpha=1$. For $x_R(t_i+\delta)=W^\top x(t_i+\delta)$ the set $\mathcal{X}(W^\top x(t_i+\delta), \|x(t_i+\delta)-Vx_R(t_i+\delta)\|)$ at the sampling instant $t_i+\delta$ is not contained in the set $\mathcal{X}(\bar{x}_R(t_i+\delta;t_i), \overline{\Delta}(t_i+\delta;t_i))$ predicted at the sampling instant t_i. This may violate feasibility of the optimization problem at the sampling instant $t_i+\delta$ as visualized by the state constraint $x_1\le 1$.

Therefore, the constraints (12) are satisfied for all $(x_R, \Delta)\in\Omega$ if

$$\sqrt{\gamma_1}\left\|c_k^\top\begin{bmatrix}V\\-K_\Omega\end{bmatrix}P_\Omega^{-1/2}\right\|+\gamma_2\left\|c_k^\top\begin{bmatrix}I\\-K_e\end{bmatrix}\right\|\le d_k,\quad k=1,\ldots,n_C. \tag{18}$$

Since the equilibrium is in the interior of the constraint set, $d_k>0$. Thus, (18) determines a set for γ_1 and γ_2 containing $\gamma_1=\gamma_2=0$ in the interior. Within this set, (17) can always be satisfied by choosing γ_1 small enough. Altogether, parameters $\gamma_1>0$ and $\gamma_2>0$ satisfying both, (17) and (18), always exist. Consequently, statement *(ii)* follows by choosing $\gamma_1>0$ and $\gamma_2>0$ sufficiently small.

(iii) Using $\dot{x}_R(t)=A_Rx_R(t)+B_Ru_R(t)$ and the algebraic Riccati equation (16) results in $\dot{E}_R(x_R)+F_R(x_R,-K_\Omega x_R)=0$. □

6.3. Optimization problem and recursive feasibility of the model predictive control scheme

In this subsection, we first derive the optimization problem of the MPC scheme using the ROM and the error bounding system such that recursive feasibility is obtained. Recursive feasibility denotes that feasibility at the initial sampling instant implies feasibility at all future sampling instants. At the end of this subsection recursive feasibility is proven based on the results of Lemma 1.

In general, feasibility at the sampling instant $t_{i+1}=t_i+\delta$ is proven by defining a candidate solution based on the solution at the sampling instant t_i. Therefore, the solution of the sampling instant t_i has to be feasible for $t\in[t_i+\delta, t_i+T]$. Then, this solution is extended for $t\in[t_i+T, t_i+\delta+T]$ based on the terminal controller.

In Problem 1, for the initial condition of the ROM the projection of the detailed state is used, i.e., $x_R(t_i+\delta)=W^\top x(t_i+\delta)$. For this choice, in general the initial condition of the ROM and the prediction are not equal. Consequently, the set of possible detailed states at the sampling instant $t_i+\delta$ might not be contained in the set predicted at the sampling instant t_i, even for α=1, as visualized in Fig. 2. Thus, this choice of the initial state might jeopardize recursive feasibility of the MPC scheme. One possibility to get recursive feasibility would be to use the initial state of the ROM as additional decision variable, which is also used in the area of robust MPC in order to deal with a model plant mismatch [18]. In order to have fewer additional decision variables, we restrict $x_R(t_i+\delta)$ to $\mathcal{I}=\{x\,|\,x=\lambda_1W^\top x(t_i+\delta)+\lambda_2\bar{x}_R(t_i+\delta;t_i)\}$ with $\lambda_1,\lambda_2\in\mathbb{R}$, i.e., the subspace containing the origin, the projection of the detailed state, and the predicted state of the ROM. Allowing for $x_R(t_i+\delta)=\bar{x}_R(t_i+\delta;t_i)$ is exploited in order to prove recursive feasibility later on. The projection of the detailed state is included in $\mathcal{I}$ to make a feedback of the measured detailed state possible. Why the subspace also contains the origin will become clear later in Section 6.5. Furthermore, having the origin as possible initial condition simplifies the proof of asymptotic stability.

For $\alpha>1$, at each sampling instant the initial error bound in (9) is overestimated by the factor α. To reduce the conservatism of initializing the error bound we set

$$\overline{\Delta}(t_i+\delta;t_i+\delta)=\min\left(\alpha\|x(t_i+\delta)-Vx_R(t_i+\delta)\|, \overline{\Delta}(t_i+\delta;t_i)\right)$$

if $x_R(t_i+\delta)=\bar{x}_R(t_i+\delta;t_i)$. Thus, we use either (9) or the prediction from the previous sampling instant. Furthermore, we use this initialization of Σ_Δ to establish recursive feasibility. Beforehand, we summarize the adjustments by stating the finite horizon optimal control problem underlying the new MPC scheme using the ROM and the error bounding system.

Problem 2. (Optimization problem for MPC using the ROM and error bounding system)

$$\underset{\bar{u}_R\in\mathcal{PC}([t_i,t_i+T],\mathbb{R}^m),\,x_R(t_i)\in\mathcal{I}}{\text{minimize}}\quad J_R(x(t_i),\bar{u}_R)$$

in which

$$J_R(x(t_i),\bar{u}_R)=\int_{t_i}^{t_i+T}F_R(\bar{x}_R(t;t_i),\bar{u}_R(t))\,dt+E_R(\bar{x}_R(t_i+T;t_i)),$$

subject to

$$\dot{\bar{x}}_R(t;t_i)=A_R\bar{x}_R(t;t_i)+B_R\bar{u}_R(t), \tag{19a}$$

$$\bar{x}_R(t_i;t_i)=x_R(t_i), \tag{19b}$$

$$c_k^\top\begin{bmatrix}V\bar{x}_R(t;t_i)\\ \bar{u}_R(t)\end{bmatrix}\le d_k-\overline{\Delta}(t;t_i)\left\|c_k^\top\begin{bmatrix}I\\-K_e\end{bmatrix}\right\|,\quad k=1,\ldots,n_C, \tag{19c}$$

$$\left(\bar{x}_R(t_i+T;t_i), \overline{\Delta}(t_i+T;t_i)\right)\in\Omega, \tag{19d}$$

$$\dot{\overline{\Delta}}(t;t_i)=-\beta\overline{\Delta}(t;t_i)+\alpha\|(I-VW^\top)(AV\bar{x}_R(t;t_i)+B\bar{u}_R(t))\|, \tag{19e}$$

$$\overline{\Delta}(t_i;t_i)=\begin{cases}\min(\alpha\|x(t_i)-Vx_R(t_i)\|, \overline{\Delta}(t_i;t_i-\delta)), & \text{if } x_R(t_i)=\bar{x}_R(t_i;t_i-\delta),\\ \alpha\|x(t_i)-Vx_R(t_i)\|, & \text{otherwise,}\end{cases} \tag{19f}$$

for all $t\in[t_i, t_i+T]$.

Problem 2 contains, in addition to the ROM (19a) and (19b), the error bounding system (19e) and (19f). If the order of Σ_D is reduced significantly, one can expect that the computational efficiency is increased by replacing Σ_D with Σ_R and the scalar Σ_Δ. To have the advantage of increased computational efficiency while guaranteeing constraint satisfaction, the constraints are tightened according to the error bound in (19c) resulting in a loss of performance and in general in a smaller region of attraction. Thus, there is a trade-off between computational efficiency and conservatism.

Due to (19f), Problem 2 does not only depend on $x(t_i)$, but also on $\bar{x}_R(t_i;t_i-\delta)$ and $\overline{\Delta}(t_i;t_i-\delta)$. For $t_i=0$ no previous solution exists and we set $\overline{\Delta}(0;0)=\alpha\|x(0)-Vx_R(0)\|$.

The input trajectory which solves Problem 2 is denoted by $u^*_{\Delta,R}(t)$, with associated state trajectory $x^*_{\Delta,R}(t;t_i)$ and error bound $\Delta^*(t;t_i)$ for $t\in[t_i, t_i+T]$. We define the cost associated with the optimal trajectories by $J^*_\Delta(x(t_i))$.

Table 1
MPC schemes considered in the present work.

MPC scheme	prediction model	$u(t)$	optimization problem
$\mathcal{D}$-MPC	Σ_D, see (1)	$u_{\mathcal{D}}(t)$, see (6)	Problem 1 for $V=W=I$
$\mathcal{R}$-MPC	Σ_R, see (4)	$u_{\mathcal{R}}(t)$, see (6)	Problem 1
Δ-MPC	Σ_R and Σ_Δ, see (4), (9)	$u_\Delta(t)$, see (20)	Problem 2

The optimal trajectories for $t \in [t_i, t_i+\delta)$ are denoted by $u_{\Delta,R}(t) = u^*_{\Delta,R}(t), x_{\Delta,R}(t) = x^*_{\Delta,R}(t;t_i)$, and $\Delta(t) = \Delta^*(t;t_i)$. These variables are used within the controller for the MOR error to compute the input of the detailed system,

$$u_\Delta(t) = u_{\Delta,R}(t) - K_e(x(t) - Vx_{\Delta,R}(t)), \quad t_i \le t < t_i + \delta, \tag{20}$$

see Fig. 1. Similar to the previously defined $u_{\mathcal{D}}$ and $u_{\mathcal{R}}$, the subscript Δ denotes, that Σ_Δ is used (together with Σ_R) within the underlying MPC scheme. According to the nomenclature for the input, we denote the MPC schemes with $\mathcal{D}$-, $\mathcal{R}$-, and Δ-MPC as summarized in Table 1.

Repeatedly applying $u_\Delta(t)$ until the next sampling instant, than measuring the state of the detailed system and solving Problem 2 at the sampling instants $t_i = i\delta$, $i \in \mathbb{N}_0$ defines the variables $u_{\Delta,R}(t)$, $x_{\Delta,R}(t)$, $\Delta(t)$, and $u_\Delta(t)$ for $t \in [0, \infty)$.

Remark 1. (Partial state measurement)
For Δ-MPC, the detailed state is required for the error feedback and in Problem 2 to initialize the error bounding system. Suppose $V^\top V = I$. Then, $V(V^\top x(t_i) - x_R(t_i)) \perp (I - VV^\top)x(t_i)$ and consequently

$$\begin{aligned}\|x(t_i) - Vx_R(t_i)\|^2 &= \|V(V^\top x(t_i) - x_R(t_i)) + (I - VV^\top)x(t_i)\|^2 \\ &= \|V(V^\top x(t_i) - x_R(t_i))\|^2 + \|(I - VV^\top)x(t_i)\|^2 \\ &= \|V^\top x(t_i) - x_R(t_i)\|^2 + \|(I - VV^\top)x(t_i)\|^2.\end{aligned}$$

Furthermore, suppose an error feedback of the form $u(t) = u_R(t) - K_R V^\top(x(t) - Vx_R(t))$ is used and $A - BK_R V^\top$ is Hurwitz. Then, a state measurement for the subspace spanned by the columns of V together with an upper bound for the MOR error $\|(I - VV^\top)x(t_i)\|$ is sufficient for Δ-MPC.

The closed loop with our proposed model predictive controller (20) leads to recursively feasible optimization problems as stated in the following theorem.

Theorem 2. *(Recursive feasibility of Δ-MPC)*
Consider the detailed system Σ_D, the ROM Σ_R defined by V, W, and the error bounding system Σ_Δ with feedback matrix K_e. Suppose that

- *Assumptions 1–5 hold,*
- *the terminal set Ω of the form (14) and the terminal cost $E(x_R) = x_R^\top Q_\Omega x_R$ with $\gamma_1 > 0$, $\gamma_2 > 0$, $P_\Omega \succ 0$, Q_Ω, and K_Ω are chosen such that (i)–(iii) in Lemma 1 hold, and*
- *the input of the detailed system is given by $u_\Delta(\cdot)$ in (20).*

Then, feasibility of Problem 2 at the sampling instant $t_0 = 0$ implies feasibility at all subsequent sampling instants $t_i = i\delta$, $i \in \mathbb{N}_0$.

Proof. The proof is partly based on standard arguments in MPC, which can be found, e.g., in [25,26,32].

In order to show feasibility for all $t_i = i\delta$, $i \in \mathbb{N}_0$ it suffices to show that from feasibility at time t_i follows feasibility at time $t_i + \delta$. Then, the statement of the theorem follows by induction.

Assume that a feasible solution $x_R(t_i)$, $\overline{u}_R(\cdot)$, $\overline{x}_R(\cdot;t_i)$, $\overline{\Delta}(\cdot;t_i)$ of Problem 2 at time t_i is given. We consider the solution candidate at time $t_i + \delta$ determined by (19a), (19e), $\tilde{x}_R(t_i+\delta) = \overline{x}_R(t_i+\delta;t_i)$, $\tilde{\Delta}(t_i+\delta) = \min\left(\alpha\|x(t_i+\delta) - V\tilde{x}_R(t_i+\delta)\|, \overline{\Delta}(t_i+\delta;t_i)\right)$, and

$$\tilde{u}_R(t) = \begin{cases} \overline{u}_R(t), & t \in [t_i+\delta, t_i+T) \\ -K_\Omega \tilde{x}_R(t), & t \in [t_i+T,\ t_i+\delta+T]. \end{cases} \tag{21}$$

The solution candidate satisfies (19f). For all $t \in [t_i+\delta,\ t_i+T)$, we have $\tilde{x}_R(t) = \overline{x}_R(t;t_i)$, $\tilde{u}_R(t) = \overline{u}_R(t)$. Consequently, we have $\dot{\tilde{\Delta}}(t) - \dot{\overline{\Delta}}(t;t_i) = -\beta(\tilde{\Delta}(t) - \overline{\Delta}(t;t_i))$. Thus, we conclude from $\tilde{\Delta}(t_i+\delta) \le \overline{\Delta}(t_i+\delta;t_i)$ that $\tilde{\Delta}(t) \le \overline{\Delta}(t;t_i)$ for all $t \in [t_i+\delta,\ t_i+T)$. Thus, from feasibility of $\overline{x}_R(t_i;t_i)$, $\overline{\Delta}(t_i;t_i)$, $\overline{u}_R(\cdot)$ follows satisfaction of (19c) for $t \in [t_i+\delta,\ t_i+T)$. Applying $\tilde{u}_R(\cdot)$ drives the ROM such that $(\tilde{x}_R(t_i+T),\quad \tilde{\Delta}(t_i+T)) \in \Omega$. For $t \in [t_i+T,\ t_i+\delta+T]$, the candidate input is given by the terminal controller and thus (19c) and (19d) are satisfied according to Lemma 1. To conclude the feasibility of Problem 2 at time $t_i+\delta$, we note that the objective for the candidate solution is finite. □

6.4. Asymptotic stability for the detailed system

With the results of Theorem 2 and Lemma 1, we are ready to state our main result concerning asymptotic stability of the detailed system controlled with the MPC scheme using the ROM and error bound.

Theorem 3. *(Asymptotic stability of Δ-MPC)*
Consider the detailed system Σ_D, the ROM Σ_R defined by V, W, and the error bounding system Σ_Δ with feedback matrix K_e. Suppose that

- *Assumptions 1–5 hold,*
- *the terminal set Ω of the form (14) and the terminal cost $E_R(x_R) = x_R^\top Q_\Omega x_R$ with $\gamma_1 > 0$, $\gamma_2 > 0$, $P_\Omega \succ 0$, Q_Ω, and K_Ω are chosen such that (i)-(iii) in Lemma 1 hold, and*
- *Problem 2 is feasible at the sampling instant $t_0 = 0$.*

Then,

(i) *the origin of the closed loop of the detailed system with the model predictive controller u_Δ given by (20) is asymptotically stable while satisfying the state and input constraints (2).*
(ii) *The region of attraction of the closed loop is given by $\mathcal{F}_\Delta$, where $\mathcal{F}_\Delta \subseteq \mathbb{R}^n$ denotes the set of all initial states $x(t_0)$ for which Problem 2 is feasible.*

Proof.

(i) First, we show asymptotic stability of the closed loop with the detailed system. Feasibility for all times $t_i = i\delta$, $i \in \mathbb{N}_0$ is guaranteed by feasibility at $t_0 = 0$ and Theorem 2.

Consider the null decision variables $x^*_{\Delta,R}(t_0;t_0) = 0$, $u^*_{\Delta,R}(\cdot;t_0) \equiv 0$, which results in $x^*_{\Delta,R}(t;t_0)^\top P_\Omega x^*_{\Delta,R}(t;t_0) = 0$ for all $t \in [t_0,\ t_0+T]$. Associated with this choice, the error bound has, by (19e), the dynamics $\dot{\Delta}^*(t;t_0) = -\beta\Delta^*(t;t_0)$ resulting in $\Delta^*(t;t_0) = \Delta^*(t_0;t_0)\exp(-\beta(t-t_0))$ for all $t \in [t_0,\quad t_0+T]$. Furthermore, consider an arbitrary initial condition $x(t_0)$ with $\|x(t_0)\| \le \gamma_2/\alpha$. Then, $\Delta^*(t_0;t_0) = \alpha\|x(t_0) - Vx^*_{\Delta,R}(t_0;t_0)\| = \alpha\|x(t_0)\| \le \gamma_2$. Consequently, since $\beta > 0$, $\Delta^*(t;t_0) \le \gamma_2 \exp(-\beta(t-t_0)) \le \gamma_2$. Since Ω is of the form (14), we have $\left(x^*_{\Delta,R}(t;t_0), \Delta^*(t;t_0)\right) \in \Omega$ for all $t \in [t_0, t_0+T]$. Thus, (19c) and (19d) hold, showing that this choice of decision variables is a feasible solution if $\|x(t_0)\| \le \gamma_2/\alpha$. Furthermore, $J^*_\Delta(x(t_0)) = 0$ for this choice of decision variables and $J_R(x(t_0), \overline{u}_R) > 0$ if $\overline{x}_R(t_0;t_0) \neq 0$ or $\overline{u}_R(\,\cdot\,;t_0) \neq 0$. Altogether, for all $\|x(t_0)\| \le \gamma_2/\alpha$ the optimal solution of Problem 2 is $x^*_{\Delta,R}(t_0;t_0) = 0$, $u^*_{\Delta,R}(\,\cdot\,;t_0) \equiv 0$.

Using the candidate solution from the proof of Theorem 2, we know that for all subsequent sampling instants $t_i > t_0$ we have $x^*_{\Delta,R}(\,\cdot\,;t_i) \equiv 0$, $u^*_{\Delta,R}(\,\cdot\,;t_i) \equiv 0$. Thus, for all $\|x(t_0)\| \le \gamma_2/\alpha$, the closed loop is described by

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu_\Delta(t)\\ &= (A - BK_e)x(t) + B(u_{\Delta,R}(t) + K_eVx_{\Delta,R}(t))\\ &= A_e x(t).\end{aligned}$$

From Assumption 5 we know that A_e is Hurwitz. Thus, the equilibrium $x=0$ for the closed loop of the detailed system with the model predictive controller $u_\Delta(\cdot)$ given by (20) is (locally) asymptotically stable.

Furthermore, the state and input constraints (2) for the detailed system are satisfied by every feasible solution of Problem 2. To this end, we observe that (19e) and (19f) together with Theorem 1 guarantees $x(t) \in \mathcal{X}(x^*_{\Delta,R}(t), \Delta^*(t))$ for all $t \in [t_i, t_i + T]$. Thus, according to Proposition 1, (2) is guaranteed by (19c). Consequently, (2) is satisfied when applying $u_\Delta(\cdot)$ to the detailed system.

(ii) For the region of attraction, we first show that for all $x(0) \in \mathcal{F}_\Delta$ the $\mathcal{L}_2$-norm of the input to the closed loop

$$\dot{x}(t) = A_e x(t) + B(u_{\Delta,R}(t) + K_eVx_{\Delta,R}(t)) \tag{22}$$

given by $\|u_{\Delta,R}(\,\cdot\,) + K_eVx_{\Delta,R}(\,\cdot\,)\|_{\mathcal{L}_2}$ is finite. With *(iii)* of Lemma 1 and the candidate solution $\tilde{x}_R$, $\tilde{\Delta}$, $\tilde{u}$, it follows along the lines of [32, proof of Lemma 3] that for all $\tau \in (0, \delta]$

$$J^*_\Delta(x(t_i+\tau)) - J^*_\Delta(x(t_i)) \le -\int_{t_i}^{t_i+\tau} F_R(x_{\Delta,R}(t), u_{\Delta,R}(t))dt\,. \tag{23}$$

In discrepancy to [32], due to the optimization of $x_R(t_i)$ in Problem 2, $x_{\Delta,R}(t)$ and thus $J^*_\Delta(x(t))$ is only piece-wise continuous. Since $J^*_\Delta(x(t))$ is a non-increasing function of time, inequality (23) holds nevertheless. By induction and non-negativity of $J^*_\Delta(x(t_i))$ for $t_i \to \infty$ follows

$$\int_0^\infty F_R(x_{\Delta,R}(t), u_{\Delta,R}(t))dt \le J^*_\Delta(x(0)). \tag{24}$$

Furthermore, since Q_R and R are positive definite, there exists a finite constant $c > 0$ such that

$$\|u_{\Delta,R}(t)\|^2 + \|K_eVx_{\Delta,R}(t)\|^2 \le \frac{c}{2}F_R(x_{\Delta,R}(t), u_{\Delta,R}(t)). \tag{25}$$

Thus, using the triangle inequality,

$$\begin{aligned}\|u_{\Delta,R}(\,\cdot\,) + K_eVx_{\Delta,R}(\,\cdot\,)\|^2_{\mathcal{L}_2} &\le 2\int_0^\infty \|u_{\Delta,R}(t)\|^2 + \|K_eVx_{\Delta,R}(t)\|^2 dt\\ &\overset{(25)}{\le} c\int_0^\infty F_R(x_{\Delta,R}(t), u_{\Delta,R}(t))dt\\ &\overset{(24)}{\le} cJ^*_\Delta(x(0)).\end{aligned}$$

From feasibility at $t_i = 0$, we have that $J^*_\Delta(x(0))$ is finite and therefore $\|u_{\Delta,R}(\,\cdot\,) + K_eVx_{\Delta,R}(\,\cdot\,)\|_{\mathcal{L}_2}$ is also finite.

Since A_e is Hurwitz, (22) is finite gain $\mathcal{L}_2$ stable [33]. Thus, $\|x(\,\cdot\,)\|_{\mathcal{L}_2}$ is finite. Furthermore, from Assumption 1 we conclude that both $x(t)$ and $u_{\Delta,R}(t) + K_eVx_{\Delta,R}(t)$ are bounded. Thus, by (22), $\frac{d}{dt}\|x(t)\|^2 = 2x^\top(t)\dot{x}(t)$ is bounded. Hence, $\|x(t)\|^2$ is uniformly continuous in t. Finally, by means of Barbalat's lemma [33, Lemma 4.2], $\lim_{t\to\infty}\|x(t)\|^2 = 0$. Consequently, $\lim_{t\to\infty} x(t) = 0$.

Thus, $\mathcal{F}_\Delta$ is a subset of the region of attraction. If $x(0) \notin \mathcal{F}_\Delta$, then $u_\Delta(t)$ is undefined and $x(0)$ is not in the region of attraction. Altogether, $\mathcal{F}_\Delta$ and the region of attraction coincide. □

According to Theorem 3, the prediction horizon T should be chosen large enough such that Problem 2 is feasible at time $t=0$. In particular, as the error dynamics are not controllable, a lower bound for the prediction horizon is determined by the initial error bound, the decay rate of the error bound, and the terminal constraint. Thus, the choice of T strongly influences the region of attraction. Clearly, other parameters such as the error feedback also influence the region of attraction as will be shown in Section 7.

6.5. *Linear-quadratic regulator as error feedback*

We consider the special case that the error feedback is chosen as the LQR of Σ_D denoted with K_D^{LQR}. Then, Δ-MPC is locally optimal as stated in the following proposition.

Proposition 2. *(LQR as error feedback)*
In addition to the assumptions of Theorem 3, assume the LQR is used as error feedback and in Problem 2 no input, state, and terminal constraints are active for $x(t)$ at $t=t_i$. Then, for all $t \ge t_i$ the input $u_\Delta(t)$ is equal to the input minimizing the infinite horizon cost $J^D_\infty(x(t_i), u)$ for the detailed system.

Proof. First, note that $A_e = A - BK_D^{\text{LQR}}$ is Hurwitz and thus there exist $\alpha \ge 1$ and $\beta > 0$ such that $\|\exp(A_e t)\| \le \alpha\exp(-\beta t)$.

Since no constraints are active at $t=0$, the null decision variables $x^*_{\Delta,R}(t_0;t_0) = 0$, $u^*_{\Delta,R}(\,\cdot\,;t_0) \equiv 0$ are one solution of Problem 2. Then, similar to the proof of Theorem 3, for all $t_j \ge t_i$ a feasible and also optimal solution is $x^*_{\Delta,R}(\,\cdot\,;t_j) \equiv 0$, $u^*_{\Delta,R}(\,\cdot\,;t_j) \equiv 0$. Thus, for all $t \ge t_i$ the input to the detailed system is $u_\Delta(t) = -K_D^{\text{LQR}}x(t)$.

For the infinite horizon optimal control problem the state and input constraints (2) are not active for $x(t_i)$. Otherwise, the constraint would be also active for the proposed MPC scheme. Thus, the input minimizing the cost $J^D_\infty(x(t_i), u)$ is determined by the LQR which coincides with the solution of the proposed MPC scheme. □

Thus, $x_{\Delta,R}(\cdot)$ and $u_{\Delta,R}(\cdot)$ are only used to fulfill constraints. Consequently, the model predictive controller may be interpreted as a disturbance from the optimal solution of the unconstrained case $u(t) = -K_D^{\text{LQR}}x(t)$ in order to satisfy the state, input, and terminal constraints. Therefore, we allow for the origin as possible initial condition in Problem 2.

7. Examples

In this section, we compare the proposed Δ-MPC scheme using the ROM (4), error bounding system (9), and error feedback K_e (7) based on Problem 2 with two MPC schemes $\mathcal{D}$- and $\mathcal{R}$-MPC based on Problem 1, see Table 1.

In Section 7.1, we consider a parameter dependent two-dimensional detailed system. The parameter will later be used to assess the influence of a varying MOR error. This small system facilitates the understanding and allows to evaluate the stability, performance with respect to the objective functional, and the size of the region of attraction. Thereafter, in Section 7.2, the computational efficiency gained by Δ-MPC and the introduced conservatism are assessed using a model of a tubular reactor.

The implementation is based on MATLAB R2012b, the Multi-Parametric Toolbox [34] and the MATLAB routine `nmpc` [35] which uses the MATLAB function `fmincon` for optimization. The constraints are only enforced at the sampling instants.

7.1. Illustrative academic example

We consider the system

$$\dot{x}(t) = \begin{bmatrix} 0.1 & 0.75\theta \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} 0.2 \\ -0.2\,\theta \end{bmatrix} u(t), \quad x(0) = x_0, \tag{26}$$

with the parameter $\theta \in [0, 1]$. The control objective is given by $Q = I$ and $R = 1$. The constraints are $|x_1(t)| \le 1$, $|x_2(t)| \le 10$, and $|u(t)| \le 0.5$. Hence, the terminal set is dominated by the constraint on u.

To keep the unstable mode in the ROM, we choose $V = W = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top$ resulting in

$$\dot{x}_R(t) = 0.1x_R(t) + 0.2u_R(t), \quad x_R(0) = x_{R,0}. \tag{27}$$

This choice of V and W also implies, that x_2 is assumed negligible. Since the influence of x_2 on x_1 increases with θ, also the effect of the MOR error will increase. Since the ROM is independent of θ, also the optimization problem of $\mathcal{R}$-MPC is independent of θ. In contrast, the varying influence of the MOR error is taken into account within Δ-MPC by Σ_Δ and the terminal region.

The first step in the design of the proposed MPC-scheme is the choice of an appropriate feedback K_e for the MOR error. We restrict ourselves to $K_e = K_D^{\text{LQR}}$. For the error bounding system we need a bound for the norm of the matrix exponential. For simplicity, we set $\alpha = 1$ so that the decay rate is given by the initial growth rate $\beta = 0.5\,\lambda_{\max}(A - BK_D^{\text{LQR}} + (A - BK_D^{\text{LQR}})^\top)$. Hence, the error bound is given by

$$\dot{\Delta}(t) = -\beta\Delta(t) + 0.2\theta|u_R(t)|, \quad \Delta(0) = \|x_0 - Vx_{R,0}\|. \tag{28}$$

Thus, the dependence of the input matrix on the parameter results in a residual proportional to θ.

For the proposed MPC-scheme, a terminal set of the form (14) with the LQR as terminal controller is used. Since $n_R = 1$, we do not limit the terminal set by choosing $P_\Omega = 1$. Thus, γ_1, γ_2 have to be determined such that (17) and (18) are satisfied resulting in

$$0.2\sqrt{\gamma_1}\,\left|\theta K_R^{\text{LQR}}\right| \le \beta\gamma_2, \tag{29a}$$

$$\sqrt{\gamma_1} + \gamma_2 \le 1, \tag{29b}$$

$$\sqrt{\gamma_1}\,\left|K_R^{\text{LQR}}\right| + \gamma_2\left\|K_D^{\text{LQR}}\right\| \le 0.5\,. \tag{29c}$$

Due to the last two inequalities, there is a compromise between a large γ_1 (terminal set for x_R) and a large γ_2 (terminal set for Δ). We set γ_1 to the largest possible value such that there exist a γ_2 satisfying (29), as this results in a larger region of attraction for almost all parameters $\theta \in [0, 1]$.

For $\mathcal{D}$- and $\mathcal{R}$-MPC as terminal set the maximal positively invariant set for the closed loop with the LQR is used and the terminal cost is $E(x) = x^\top Q_\Omega x$ where Q_Ω is the solution of the ARE (16) associated with (26) or (27). To implement the three MPC schemes a sampling time of 0.2, a prediction horizon of 10, and a piece-wise constant input u_R are used.

Trajectories of the closed loop are shown in Fig. 3 for $\theta = 0.95$. As initial condition the maximal x_1 for $x_2 = 0$ in the region of attraction of Δ-MPC is taken resulting in $x_0 = [0.1768\ 0]^\top$. Despite the large θ and hence influence of the discarded state x_2, the trajectories of $\mathcal{D}$- and Δ-MPC are almost identical. For $t < 2.0$, the input constraint is active and Δ-MPC uses the initial condition of the ROM to fulfill the constraints. Since $u_{\Delta,R} = 0$, the input of Σ_D originates from the error feedback K_e. Furthermore, the input constraint is enforced by $\Delta(t)\,\|\,K_e\,\| \le 0.5$. Hence, the conservatism introduced by the error bound results in $u_\Delta > -0.5$ for $t \approx 0$. For $t \ge 2.0$, the initial

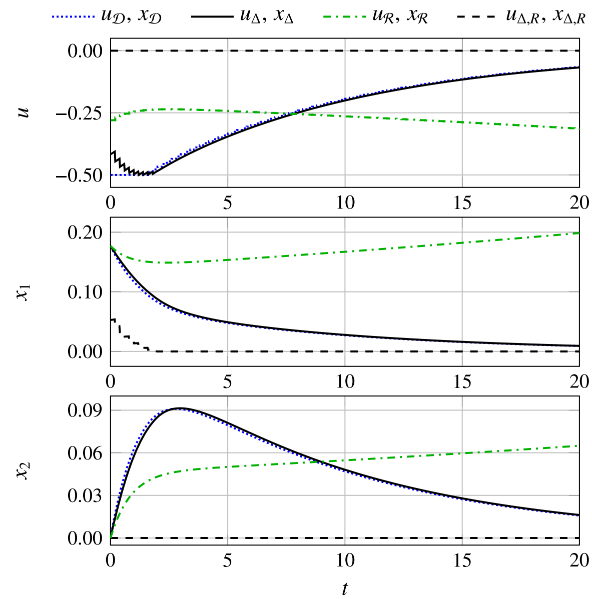

Fig. 3. Trajectories of the closed loop with MPC using the detailed system ($\mathcal{D}$-MPC), ROM and error bound (Δ-MPC) and ROM ($\mathcal{R}$-MPC) for $\theta = 0.95$. To judge the influence of the error feedback and the input of the ROM to the overall input of Δ-MPC $u_\Delta(t) = u_{\Delta,R}(t) - K_e(x_\Delta(t) - Vx_{\Delta,R}(t))$, also $u_{\Delta,R}(t)$ and the estimated states $Vx_{\Delta,R}(t)$ are shown.

condition and input of the ROM is 0 and thus the optimal input $u(t) = -K_D^{\text{LQR}}x(t)$ is applied as discussed in Section 6.5.

Due to the large MOR error, $\mathcal{R}$-MPC diverges while the proposed MPC approach drives the detailed system to the origin. This shows that the MOR error is relevant in this example and should be accounted for in the control design.

7.1.1. Stability

The most fundamental distinction between the proposed MPC scheme and $\mathcal{R}$-MPC is the guaranteed asymptotic stability. Fig. 3 visualizes that $\mathcal{R}$-MPC results in an unstable closed loop for $\theta = 0.95$. In fact, this holds for $\theta \approx 1$. Using the feedback K_e within $\mathcal{R}$-MPC may lead to asymptotic stability. Unfortunately, due to the dependence of $u(t)$ on the error $e(t)$, then even pure input constraints cannot be guaranteed.

7.1.2. Performance

Fig. 3 suggests that $\mathcal{D}$-MPC and Δ-MPC result in an almost identical performance for this initial condition and $\theta = 0.95$. In this subsection, we evaluate the performance of the three MPC schemes for several values of θ and varying $x_1(0)$. Since x_2 is assumed negligible, we set $x_2(0) = 0$. This choice does not imply that $x_2(\cdot) \equiv 0$ as can be seen in Fig. 3.

As performance measure we take the integrated stage cost of the closed loop with Σ_D for $t \in [0, 50]$ in addition to the optimal cost of the unconstrained Σ_D for $t \in [50, \infty]$. We denote this overall cost with J_i^D in which $i \in \{\mathcal{D}, \Delta, \mathcal{R}\}$ indicates the utilized MPC scheme. For $\mathcal{R}$-MPC, the optimization problem may be feasible at $t_0 = 0$ but infeasible for $t \in (0, 50]$. Then, we set $J_{\mathcal{R}}^D / x(0)^\top P_D x(0) = 7$ in Fig. 4a. For $\theta \le 0.5$ and thus small influence of the MOR error, the performance of all three MPC schemes is comparable. However, for increasing θ, the performance of $\mathcal{R}$-MPC gets worse whereas Δ-MPC still achieves a performance comparable to $\mathcal{D}$-MPC, as visualized in Fig. 4a for $\theta = 0.8$. For $\theta = 0.8$ and $x_1(0) \approx 0.5$, $\mathcal{R}$-MPC applies $u(t) = -0.5$ at the beginning. This leads to an increase of x_2 and consequently to a slow decay of x_1. Thus, $J_{\mathcal{R}}^D$ rapidly increases for $x_1(0) \approx 0.5$.

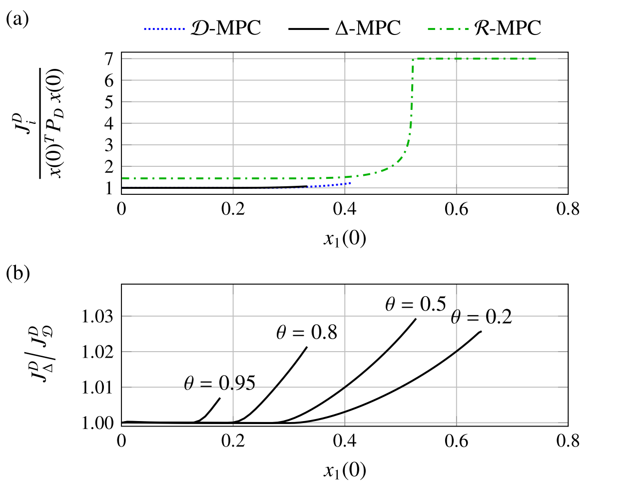

Fig. 4. Objective values for the closed loop of the detailed system and $\mathcal{D}$-MPC, Δ-MPC, and $\mathcal{R}$-MPC. (a) Objective values normalized with the optimal cost of the unconstrained Σ_D for $\theta = 0.8$. (b) Objective value of Δ-MPC relative to the objective value realized with $\mathcal{D}$-MPC for $\theta \in \{0.2, 0.5, 0.8, 0.95\}$.

Fig. 4a also depicts the set of initial conditions with feasible solutions for $x_2 = 0$. The region of attraction will be considered in Section 7.1.3. Here, we point out that $\mathcal{R}$-MPC neglects the influence of x_2 and has a feasible solution for all $x_1(0) \in [-0.74, 0.74]$ independent of θ. Thus, for $\theta = 0.8$, the set with a feasible solution is even larger than for $\mathcal{D}$-MPC. However, $\mathcal{R}$-MPC is initially feasible and infeasible at a later sampling instant for all $0.53 \le x_1(0) \le 0.74$. Thus, the larger feasible set is accompanied with the lack of guaranteed asymptotic stability.

In Section 6.5, we have shown that the proposed MPC scheme is locally optimal for the chosen error feedback. To illustrate how large the region with optimal performance is, we depict the attained performance normalized with the performance of $\mathcal{D}$-MPC in Fig. 4b. The region where Δ-MPC is optimal is reasonably large and the performance degradation is less than 3%. In contrast, $\mathcal{R}$-MPC is locally sub-optimal by 44% for $\theta = 0.8$.

7.1.3. Region of attraction

In addition to the optimality with respect to the cost functional, the size of the set of initial conditions with guaranteed asymptotic stability is another possibility to assess an MPC scheme. For $\mathcal{D}$-MPC and the proposed Δ-MPC scheme this set is given by the set of initial conditions with a feasible solution denoted by $\mathcal{F}_{\mathcal{D}}$ and $\mathcal{F}_\Delta$, respectively. The $\mathcal{R}$-MPC scheme gives no guarantee for asymptotic stability and looking at the feasible set can be misleading as indicated in Fig. 4a. Therefore, $\mathcal{R}$-MPC is not considered in this subsection.

To simplify visualization, we restrict $x_2(0)$ to 0. We denote the length of the interval for $x_1(0)$ with a feasible solution by $l_i = \left|\mathcal{F}_i|_{x_2=0}\right|, \quad i \in \{\mathcal{D}, \Delta\}$. The ratio $l_\Delta / l_{\mathcal{D}}$ is depicted in Fig. 5. For small θ, the residual and thus the error bound is small. Consequently, $l_\Delta \approx l_{\mathcal{D}}$. For increasing θ the system gets harder to control and both, l_Δ and $l_{\mathcal{D}}$, get smaller. Notably, for $\theta \le 0.83$ the feasible set for $x_1(0)$ of the proposed MPC scheme contains 80% of the feasible set of $\mathcal{D}$-MPC. Even for $\theta = 0.9$, where $\mathcal{R}$-MPC shows a poor performance, the ratio $l_\Delta / l_{\mathcal{D}}$ is above 75%. For higher parameter values, l_Δ decreases substantially, since β decreases while $\|K_e\|$ considerably increases. For $\theta \ge 0.992$, Assumption 5 is not fulfilled for $\alpha = 1$ and the LQR as error feedback. Hence, we also used controllers other than the LQR for the error feedback. Then, feasibility for $\theta \approx 1$ is achieved. Furthermore, the feasible set significantly increases for $\theta \le 0.3$ and $\theta \ge 0.9$.

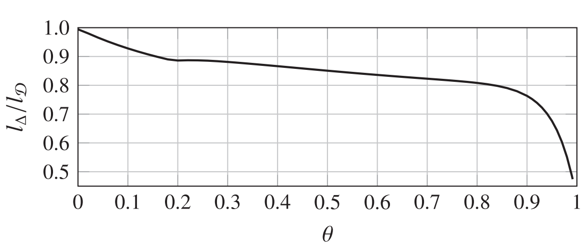

Fig. 5. Ratio of the interval for $x_1(0)$ within the region of attraction for Δ-MPC and $\mathcal{D}$-MPC. The initial condition of x_2 is set to 0.

Summarizing, we have studied a parameter dependent two-dimensional example, in which the parameter changes the MOR error. We have compared stability, performance, and region of attraction for the proposed and alternative MPC schemes in dependence of this parameter. We have seen that even for a significant MOR error, the proposed MPC scheme shows a good performance while guaranteeing asymptotic stability.

7.2. Non-isothermal tubular reactor

To evaluate the computational efficiency of the proposed MPC scheme the model of a non-isothermal tubular reactor [13] is employed. The model of the tubular reactor assumes plug-flow behavior, i.e., without diffusion or dispersion. The flow enters the reactor with a constant concentration $C_{\rm in}$ of the reactant A and temperature $T_{\rm in}$. Inside the reactor, a first order, irreversible, exothermic reaction takes place. In order to prevent undesirable byproducts, the temperature in the reactor must satisfy

$$T \le 400\,\mathrm{K}. \tag{30}$$

To control the temperature inside the reactor, the wall temperature can be changed between 280 K and 400 K by three independent heat exchangers as visualized in Fig. 6.

Due to an error in an upstream plant the inflow temperature may change from the nominal value $T_{\rm nom}$=340 K to a value within [315 K, 365 K]. The control goal is to achieve a high conversion ratio despite these changes of $T_{\rm in}$ while satisfying the temperature constraints. In the following, we assume the reactor is at the steady state for the nominal temperature of the inflow for $t < 0$ and consider step changes of $T_{\rm in}$ at $t = 0$. Later in Section 7.2.1, we define a setpoint depending on $T_{\rm in}$ and MPC is used to steer the tubular reactor to this setpoint. Therefore, the step change of $T_{\rm in}$ can be considered as a shift of the setpoint. Hence, the considered MPC schemes can be applied.

In [13,36], two MPC schemes using an ROM are discussed in order to control the tubular reactor. Both controllers in [13,36] do neither guarantee satisfaction of state constraints, nor asymptotic stability. However, the problem setup and the MOR approach presented in the following are extensively based on [13,36].

7.2.1. Problem setup

The PDE model of the tubular reactor originates from [37] and is inspired by [38]. This PDE is linearized around the steady state computed in [36]. Afterwards, the reactor is divided into 150 sections and the partial derivatives are approximated by backward difference approximations resulting in a linear ODE [36]. The modeling

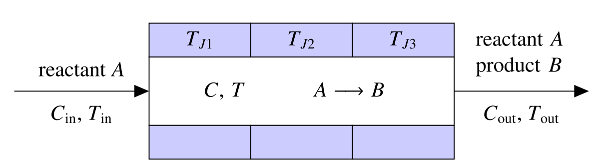

Fig. 6. Tubular reactor with 3 cooling/heating jackets. The schematic is based on [13].

errors arising from the linearization and the spatial discretization are not considered, as this work discusses the error originating from the MOR. In the following we only consider the ODE model, not the original PDE, and study the numerical efficiency of the proposed MPC scheme by this practically motivated example.

The detailed linear model

$$\dot{x}(t) = Ax(t) + B_u u(t) + B_T T_{\text{in}}(t), \quad x(0) = x_0$$

consists of 303 states, 150 for the temperature inside the reactor, 150 for the concentration inside the reactor, and three for the jacket temperatures. The temperature and concentration inside the reactor are denoted with T_i and C_i, $i = 1, \ldots, 150$, with $T_{\text{out}} = T_{150}$ and $C_{\text{out}} = C_{150}$. The three control inputs are the changes of the jacket temperatures $u = [\dot{T}_{J1} \;\; \dot{T}_{J2} \;\; \dot{T}_{J3}]^\top$.

To apply the considered MPC schemes we use a setpoint $(x_{\text{SP}}, u_{\text{SP}})$ with $u_{\text{SP}} = 0$ depending on T_{in} which is determined by

$$\underset{x_{\text{SP}}}{\text{minimize}}\; 0.7\left(\frac{C_{\text{out}}}{C_{\text{in}}}\right)^2 + \frac{0.3}{150}\sum_{i=1}^{150}\left(\frac{T_i - T_{\text{nom}}}{T_{\text{nom}}}\right)^2$$

subject to

$$0 = Ax_{\text{SP}} + B_u u_{\text{SP}} + B_T T_{\text{in}}, \tag{31a}$$

$$T_i \le 395\,\text{K}, \quad i = 1, \ldots, 150, \tag{31b}$$

$$285\,\text{K} \le T_{Ji} \le 395\,\text{K}, \quad i = 1, 2, 3. \tag{31c}$$

In contrast to [13,36], our controller guarantees satisfaction of the constraints. Thus, we allow a setpoint closer to the temperature constraint (30). Furthermore, in comparison to [13,36], we choose a higher weight for the output concentration as considered in [39]. Hence, higher temperature deviations from T_{nom} occur, consequently, making the temperature constraint (30) more important. Altogether, the output concentration at the setpoint for the nominal inflow is $5.1 \cdot 10^{-4}$ mol/L and, compared to [13,36], it is decreased by a factor of three.

The quadratic stage cost weighting the difference to the setpoint is given by

$$F(x,u) = 5\left(\frac{C_{\text{out}} - C_{\text{SP,out}}}{C_{\text{in}}}\right)^2 + 10^{-5}\sum_{i=1}^{149}\left(\frac{C_i - C_{\text{SP},i}}{C_{\text{in}}}\right)^2 + 0.0023\sum_{i=1}^{150}\left(\frac{T_i - T_{\text{SP},i}}{T_{\text{nom}}}\right)^2 + 0.05\sum_{i=1}^{3}\left(\frac{T_{Ji} - T_{\text{SP},Ji}}{T_{\text{nom}}}\right)^2 + 137.5\sum_{i=1}^{3}\left(\frac{\dot{T}_{Ji}}{T_{\text{nom}}}\right)^2.$$

Thus, deviations in the output concentration are strongly penalized. The values weighting $C_{\text{out}} = C_{150}$, T_i, and $\dot{T}_{Ji}$ are derived from [13]. The other two terms weighting C_i and T_{Ji} are added due to Assumption 2 and to force the jacket temperatures to converge to the setpoint.

7.2.2. Error bound and model order reduction

For the tubular reactor, error bounds using an unweighted norm are conservative. As alternative to a weighted norm, we apply a linear state transformation to the detailed system and the objective functional. The state transformation is computed such that (10) holds with $\alpha = 1$, $\beta = 0.15$ and the LQR controller as error feedback. The state transformation is computed similar to [31] using YALMIP [40] with the solver SDPT3 [41].

The ROM is derived in the style of [13,36] and relies on the proper orthogonal decomposition [5,42]. The snapshot of state trajectories relies on simulations with $T_{\text{in}} \in [315\,\text{K}, 365\,\text{K}]$ and step changes of $u_i \in [-17\,\text{K/s}, 17\,\text{K/s}]$. To compute the eigenvectors of the snapshot we use the implementation contained in RBmatlab [43]. To have a small enough error bound and feasibility of Problem 2 at $t = 0$ we choose $n_R = 45$. Future improvements like extending the scalar error bounding system or more sophisticated model reduction methods most likely allow a smaller order of the reduced model.

For solving ODEs we use the sundialsTB [44]. One simulation with forward sensitivity analysis is on average more than 10 times faster for the ROM with the error bounding system than the detailed model.

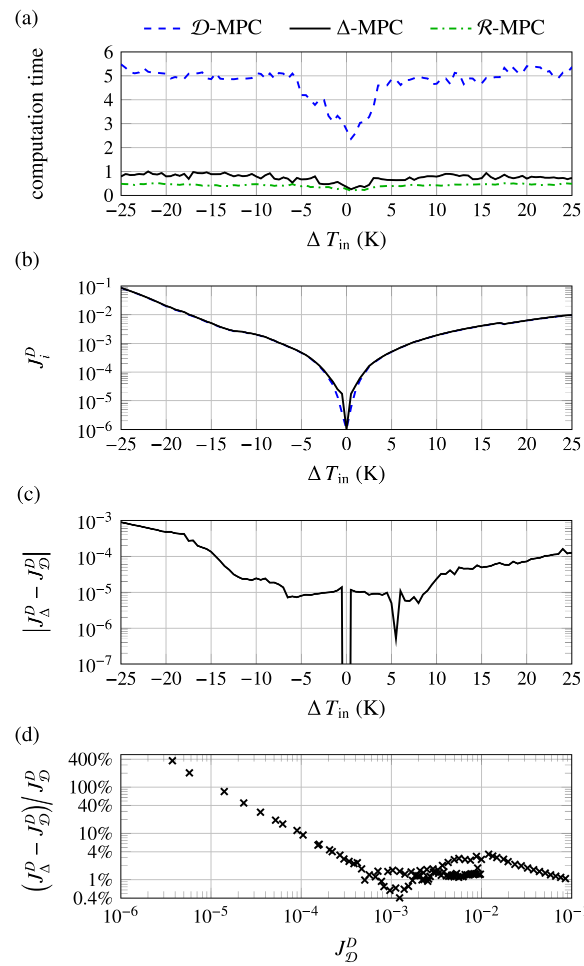

Fig. 7. Simulation results for the tubular reactor. (a) Computation times of $\mathcal{D}$-MPC, Δ-MPC, and $\mathcal{R}$-MPC for the optimization at $t = 0$. The computation times are normalized by the maximal computation time of Δ-MPC. (b) Objective value of $\mathcal{D}$- and Δ-MPC for the closed loop of the detailed model and the respective model predictive controller. (c) Absolute difference of the objective value of $\mathcal{D}$- and Δ-MPC for the closed loop of the detailed model and the respective model predictive controller. (d) Relative performance degradation over the objective value of the closed loop with $\mathcal{D}$-MPC.

7.2.3. Comparison of the model predictive control schemes

We use the same sampling time of $\delta = 0.2$ s and prediction horizon of $T = 80\,\delta$ as in [13,36]. For all three MPC schemes, the terminal cost is determined by the solution of the algebraic Riccati equation and the terminal sets are computed using the LQR as terminal controller. Since polytopic positively invariant sets are difficult or impossible to compute for the dimension of the systems, we use ellipsoids centered at the setpoint as terminal sets for $\mathcal{D}$- and $\mathcal{R}$-MPC. For Δ- and $\mathcal{R}$-MPC the volume of the ellipsoidal terminal set for x_R is maximized using YALMIP [40] with the solver SDPT3 [41]. For $\mathcal{D}$-MPC, the maximal volume ellipsoid could not be computed due to the large dimension of the system. Thus, we used an ellipsoid based on the state transformation described above.

For the optimization, we use the MATLAB routine `fmincon`. We provide the gradients of the objective function and the constraints. From the list of available optimization algorithms we choose 'active-set', as it results for this example in the lowest computation time for all MPC schemes. The initial guess of the optimal solution used by the optimization routine is based on the optimal solution of the previous sampling instant.

The computation time required by the three MPC schemes is shown in Fig. 7a. The maximal computation time is decreased by the proposed scheme by a factor of 5.5 compared to $\mathcal{D}$-MPC and increased by a factor of 2 compared to $\mathcal{R}$-MPC. Thus, the proposed MPC scheme shows a sensible compromise between increased computational efficiency and guarantees for the closed loop. The guaranteed constraint satisfaction and recursive feasibility given by Δ-MPC are also relevant for the tubular reactor, since $\mathcal{R}$-MPC violates the temperature constraint for ΔT_{in}=25 K. This underpins that $\mathcal{R}$-MPC does not guarantee recursive feasibility.

Finally, we evaluate the conservatism of Δ-MPC. Since the order of the ROM is chosen such that Problem 2 is feasible for all $\Delta T_{\text{in}} \in$ [-25 K, 25 K], the conservatism with respect to the region of attraction is irrelevant. The conservatism with respect to the performance is depicted in Fig. 7b–d. For medium to large values of the cost, i.e., $J_{\mathcal{D}}^{D} \geq 10^{-3}$, the performance of Δ-MPC is at most 4% worse than $\mathcal{D}$-MPC, which is reasonably low compared to the increase in computational efficiency. For $J_{\mathcal{D}}^{D} < 10^{-3}$, the difference of the cost values is smaller than approximately 10^{-5} and may be considered negligible. The larger objective value for Δ-MPC around $\Delta T_{\text{in}} \approx 0$ K results from the optimization of $x_R(t_i)$, since the projection of the cost functional is a bad approximation for $x_R(t_i) \not\approx W^{\top} x(t_i)$.

In summary, this simulation study shows that Δ-MPC can increase the computational efficiency significantly, compared to an MPC scheme based on the detailed model, while at the same time resulting in a small conservatism with respect to performance and the region of attraction. The small degradation of the performance is underpinned for the academic example by the comparison with $\mathcal{R}$-MPC which results in a significantly worse performance. The importance of the guaranteed satisfaction of constraints and asymptotic stability is emphasized by both examples when using only an ROM for prediction. For the academic example the closed loop diverges for large MOR errors and for the tubular reactor the constraints are violated.

8. Conclusion

We presented a novel model predictive control (MPC) scheme using a reduced order model for prediction to control a possibly unstable high-dimensional linear system subject to hard input and state constraints. To guarantee satisfaction of hard input and state constraints for the high-dimensional system despite using a reduced order model for prediction, we employed a bound for the model order reduction error. The error bound is given by a scalar ordinary differential equation which predicts the error bound. In contrast to previous results, the predicted input trajectory is taken into account within the error bound which may significantly reduce the conservatism. By using the initial state of the reduced order model as decision variable and appropriate initialization of the error bounding system, recursive feasibility is established despite the model order reduction error. Moreover, we derived constructive design conditions that ensure asymptotic stability for the closed loop of the proposed model predictive controller when applied to the high-dimensional system.

We applied the proposed MPC scheme to two examples. For an illustrative academic example, a common MPC scheme based on the reduced order model, which neglects the model order reduction error, results in an unstable closed loop. In contrast, for a large influence of the model order reduction error, the proposed MPC scheme achieves a large region of attraction and a performance comparable to the MPC scheme using the detailed system. Furthermore, the example demonstrates that by choosing the linear quadratic regulator as error feedback our approach results in an optimal solution for a large region around the equilibrium. The application of the discussed MPC schemes to a model of a tubular reactor demonstrates that the proposed MPC approach can increase the computational efficiency significantly, while resulting in an almost optimal performance and large region of attraction. Altogether, the simulation study shows that the proposed MPC approach can achieve a reasonable trade-off between computational efficiency and conservatism while guaranteeing constraint satisfaction and asymptotic stability.

Future work may extend the proposed MPC approach such that an upper bound for the objective functional is minimized similar to [22]. This results in a guarantee for the worst-case value of the objective functional and may improve the achieved performance.

The optimization problem of the proposed MPC scheme gives conditions on the reduced order model and the model order reduction error in order to satisfy constraints and guarantee asymptotic stability. This conditions might be exploited in the future to develop a method for model order reduction of constrained linear control systems.

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